

# HOSSAM GHANEM

## (6) 7.4 Integral Using Natural Logarithm and Exponential Function (A)

Differential :

$\frac{d}{dx} \log_a(x) = \frac{1}{\ln a} \cdot \frac{1}{x}$	$\frac{d}{dx} a^x = a^x \cdot \ln a$
$\frac{d}{dx} \ln x = \frac{1}{x}$	$\frac{d}{dx} e^x = e^x$

Integral:

$\int \frac{1}{x} dx = \ln x  + c$	$\int a^x dx = \frac{a^x}{\ln a} + c$	$\int e^x dx = e^x + c$
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Integral

$\int \sin x dx = -\cos x + C$	$\int \tan x dx = -\ln \cos x  + C$
$\int \cos x dx = \sin x + C$	$\int \cot x dx = \ln \sin x  + C$
$\int \sec^2 x dx = \tan x + C$	$\int \sec x dx = \ln \sec x + \tan x  + C$
$\int \csc^2 x dx = -\cot x + C$	$\int \csc x dx = \ln \csc x - \cot x  + C$
$\int \sec x \tan x dx = \sec x + C$	$\int e^t du = e^t + C$
$\int \csc x \cot x dx = -\csc x + C$	$\int \frac{1}{t} du = \ln t  + C$

**TO SOLVE**

$\int f(x) \cdot f'(x) dx$
$\int [f(x)]^n \cdot f'(x) dx$
$\int \frac{f'(x)}{f(x)} dx$
$\int e^{f(x)} \cdot f'(x) dx$
$\int \tan f(x) \cdot f'(x) dx$
$\int \cot f(x) \cdot f'(x) dx$
$\int \sec f(x) \cdot f'(x) dx$
$\int \csc f(x) \cdot f'(x) dx$



**Substitute**

Let
$t = f(x)$
$dt = f'(x) dx$



**Result**

$\int t dt$
$\int t^n dt$
$\int \frac{1}{t} dt$
$\int e^t dt$
$\int \tan t dt$
$\int \cot t dt$
$\int \sec t dt$
$\int \csc t dt$

**Example 1** Find  $\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$

**Solution**

$$I = \int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$$

$$t = e^x - e^{-x} \quad dt = (e^x + e^{-x}) dx$$

$$I = \int \frac{1}{t} dt = \ln|t| + c = \ln|e^x - e^{-x}| + c$$

**Example 2**

Evaluate the following integrals  $\int \frac{(1 + e^{-\sqrt{x}})^5}{\sqrt{x} e^{\sqrt{x}}} dx$

**Solution**

$$I = \int \frac{(1 + e^{-\sqrt{x}})^5}{\sqrt{x} e^{\sqrt{x}}} dx$$

$$t = 1 + e^{-\sqrt{x}}, \quad dt = e^{-\sqrt{x}} \cdot \frac{-1}{2\sqrt{x}}, \quad -2dt = \frac{1}{\sqrt{x} e^{\sqrt{x}}} dx$$

$$I = -2 \int t^5 dt = -2 \cdot \frac{1}{6} t^6 + c = -\frac{1}{3} (1 + e^{-\sqrt{x}})^6 + c$$

**Example 3**

6 March 1997

Evaluate the following integrals

$$\int \frac{1}{(1 + e^{-2x})} dx,$$

**Solution**

$$I = \int \frac{1}{1 + e^{-2x}} dx = \int \frac{e^{2x}}{e^{2x} + 1} dx \quad \text{بالضرب بسط و مقام في } e^{2x}$$

$$t = e^{2x} + 1 \quad dt = 2e^{2x} dx \quad \frac{1}{2} dt = e^{2x} dx$$

$$I = \frac{1}{2} \int \frac{1}{t} dt = \frac{1}{2} \ln|t| + c = \frac{1}{2} \ln(e^{2x} + 1) + c$$

**Example 4**

23 November 2007 A

Evaluate the following integrals

$$\int \frac{dx}{\sqrt{x}(1 + e^{-\sqrt{x}})}$$

**Solution**

$$I = \int \frac{dx}{\sqrt{x}(1 + e^{-\sqrt{x}})} = \int \frac{e^{\sqrt{x}}}{\sqrt{x}(e^{\sqrt{x}} + 1)} dx \quad \text{بالضرب بسط و مقام في } e^{\sqrt{x}}$$

$$\text{Let } t = e^{\sqrt{x}} + 1 \quad \rightarrow \quad dt = \frac{1}{2\sqrt{x}} e^{\sqrt{x}} dx \quad \rightarrow \quad 2 dt = \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$$

$$I = 2 \int \frac{1}{t} dt = 2 \ln|t| + c = 2 \ln(e^{\sqrt{x}} + 1) + c$$



**Example 5**

15 July 2003 A

. Evaluate the following integrals

$$\int \frac{x^{1/e} - e^{1/x}}{x^2} dx$$

**Solution**

$$I = \int \frac{x^{1/e} - e^{1/x}}{x^2} dx = \int \frac{x^{\frac{1}{e}}}{x^2} - \frac{e^{\frac{1}{x}}}{x^2} dx$$

$$I_1 = \int \frac{x^{\frac{1}{e}}}{x^2} dx = \int x^{\frac{1}{e}-2} dx = \frac{1}{\frac{1}{e}-1} x^{\frac{1}{e}-1} + c_1$$

$$I_2 = \int \frac{e^{\frac{1}{x}}}{x^2} dx = \int e^{\frac{1}{x}} \cdot \frac{1}{x^2} dx$$

$$\text{Let } t = \frac{1}{x} \quad dt = \frac{-1}{x^2} dx$$

$$I_2 = - \int e^t du = -e^t + c_2 = -e^{\frac{1}{x}} + c_2$$

$$I = \frac{1}{\frac{1}{e}-1} x^{\frac{1}{e}-1} - e^{\frac{1}{x}} + c = \frac{e}{1-e} x^{\frac{1}{e}-1} - e^{\frac{1}{x}} + c$$

**Example 6**

29 July 2009 A

. Evaluate the following integrals

$$\int \frac{1 + \ln x}{2 - 3x \ln x} dx$$

**Solution**

$$I = \int \frac{1 + \ln x}{2 - 3x \ln x} dx$$

$$t = 2 - 3x \ln x \quad dt = -3 \ln x - 3x \cdot \frac{1}{x} dx \quad dt = -3 \ln x - 3 dx$$

$$dt = -3(\ln x + 1) dx \quad \frac{-1}{3} dt = (\ln x + 1) dx$$

$$I = \frac{-1}{3} \int \frac{1}{t} dt = \frac{-1}{3} \ln|t| + c = \frac{-1}{3} \ln|2 - 3x \ln x| + c$$

**Example 7**

22 July 2007

. Evaluate the following integrals

$$\int \frac{1}{x(1 + \ln(\sqrt{x}))^{\frac{1}{3}}} dx$$

**Solution**

$$I = \int \frac{1}{x(1 + \ln(\sqrt{x}))^{\frac{1}{3}}} dx$$

$$t = 1 + \ln(\sqrt{x}) \quad t = 1 + \frac{1}{2} \ln x \quad dt = \frac{1}{2x} dx \quad 2dt = \frac{1}{x} dx$$

$$I = \int \frac{1}{(1 + \ln(\sqrt{x}))^{\frac{1}{3}}} \cdot \frac{1}{x} dx = 2 \int \frac{1}{t^{\frac{1}{3}}} dt = 2 \int t^{-\frac{1}{3}} dt = 2 \cdot \frac{3}{2} t^{\frac{2}{3}} + c = 3(1 + \ln(\sqrt{x}))^{\frac{2}{3}} + c$$

**Example 8**

27 Nov. 2008 A

Evaluate the following integrals

$$\int \frac{\ln x^3}{x(4 + 3 \ln^2 x)} dx$$

**Solution**

$$I = \int \frac{\ln x^3}{x(4 + 3 \ln^2 x)} dx = \int \frac{1}{4 + 3 \ln^2 x} \cdot \frac{\ln x^3}{x} dx$$

$$u = 4 + 3 \ln^2 x$$

$$du = 6 \ln x \cdot \frac{1}{x} dx$$

$$\frac{1}{2} du = \frac{3 \ln x}{x} dx$$

$$\frac{1}{2} du = \frac{\ln x^3}{x} dx$$

$$I = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln|u| + c = \frac{1}{2} \ln(4 + 3 \ln^2 x) + c$$

**Example 9**

29 July 2009 A

Evaluate the following integrals

$$\int_{e^{-2}}^{e^{-1}} \frac{1}{t \ln t} dt$$

**Solution**

$$I = \int_{e^{-2}}^{e^{-1}} \frac{1}{t \ln t} dt$$

$$\text{Let } u = \ln t$$

$$du = \frac{1}{t} dt$$

$$\text{at } t = e^{-2} \quad u = \ln e^{-2} = -2$$

$$t = e^{-1} \quad u = \ln e^{-1} = -1$$

$$I = \int_{-2}^{-1} \frac{1}{u} du = [\ln|u|]_{-2}^{-1} = \ln 1 - \ln 2 = -\ln 2$$

**Example 10**

Find

$$\int \frac{\sin 2x}{\sin^2 x - 1} dx$$

**Solution**

$$I = \int \frac{\sin 2x}{\sin^2 x - 1} dx$$

$$u = \sin^2 x - 1$$

$$du = 2 \sin x \cos x dx$$

$$du = \sin 2x dx$$

$$I = \int \frac{1}{u} du = \ln|u| + c = \ln|\sin^2 x - 1| + c$$



**Example 11** Find  $\int \tan x \, dx$

**Solution**

$$I = \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

$$u = \cos x$$

$$du = -\sin x \, dx$$

$$I = -\int \frac{1}{u} \, du = -\ln|u| + c = -\ln|\cos x| + c = \ln|\sec x| + c$$

**Example 12**

13 March 2001 A

Evaluate the following integrals

$$\int \frac{3 - 2 \tan x}{2 + 3 \tan x} \, dx$$

**Solution**

$$I = \int \frac{3 - 2 \tan x}{2 + 3 \tan x} \, dx = \int \frac{3 \cos x - 2 \sin x}{2 \cos x + 3 \sin x} \, dx \quad \text{بالضرب في } \cos x \text{ بسط ومقام}$$

$$\text{Let } u = 2 \cos x + 3 \sin x$$

$$du = (-2 \sin x + 3 \cos x) \, dx$$

$$du = (3 \cos x - 2 \sin x) \, dx$$

$$I = \int \frac{1}{u} \, du = \ln|u| + c = \ln|2 \cos x + 3 \sin x| + c$$

**Example 13**

3 November 1994

Evaluate the following integrals

$$\int (\tan x - 1)^2 \, dx$$

**Solution**

$$I = \int (\tan x - 1)^2 \, dx = \int (\tan^2 x + 2 \tan x + 1) \, dx = \int (\tan^2 x + 1 + 2 \tan x) \, dx = \int (\sec^2 x + 2 \tan x) \, dx$$

$$= \tan x - 2 \ln|\cos x| + c$$



## Homework

1 Evaluate the following integrals  $\int \frac{e^x}{3 + e^{x+2}} dx$  22 July 2007

2 Evaluate the following integrals  $\int \frac{1}{e^{1-x} + 1} dx$  7 July 1997

3 Evaluate the following integrals  $\int \frac{x^{e^2} - e^{\frac{1}{x^2}}}{x^3} dx$

4 Evaluate the following integrals  $\int \frac{dx}{x(\ln x)^2}$  11 October 1999

5 Evaluate the following integrals  $\int \frac{\ln x^2}{x(9 + (\ln x)^2)} dx$

6 Evaluate the following integrals  $\int \frac{\cos x \sin x}{\cos^2 x - 3} dx$  8 October 1997

7 Evaluate the following integrals  $\int \frac{\cos x \sin x}{\cos^2 x - 1} dx$

8 (3 pts) . Evaluate the integral  $\int \frac{10^{1/x} - \tan(1/x)}{x^2} dx.$  33 April 10, 2011

9 (3 pts) . Evaluate the integral  $\int \frac{\sin x}{\sqrt{8 + \sin^2 x}} dx.$  33 April 10, 2011